I. Introduction

Orthogonal functions, to be defined below, are extensively utilized in Physics and related science and engineering disciplines. In a sense, the role of orthogonal functions is similar to that of unit vectors in 2, 3, or more dimensions. Indeed, the way an arbitrary vector, in three dimensions (3-D), can be written as a linear combination of the orthogonal basis vectors (i.e., unit vector \( \hat{i}, \hat{j}, \text{and } \hat{k} \) ),

\[
\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \quad (3-D) \text{ or } \vec{v} = \sum_{i=1}^{N} v_i \hat{i}, \text{ in } N \text{ dimensions (N-D)}
\]

is the same way in which an arbitrary function \( G(x) \) that satisfies certain conditions can be written as

\[
G(x) = \sum_{n=1}^{\infty} a_n f_n(x), \text{ where } f_1(x), f_2(x), \ldots \text{ are orthogonal functions.}
\]

Please note, from the onset, that the expansion of \( G(x) \) in terms of orthogonal functions generally entails an infinite set of orthogonal functions \( f_1(x), f_2(x), \ldots, f_n(x), \ldots \), unlike the case of the expansion of a vector \( \vec{v} \) where the set of orthogonal unit vectors is generally finite.

II. Definition

Functions \( f_1(x), f_2(x), \ldots, f_n(x) \) are said to be orthogonal—over the interval \( [a, b] \) and with the density or weighting function \( w(x) \)—if and only if

\[
\forall i, \forall j, \int_{a}^{b} f_i^*(x)f_j(x)w(x)dx = N_{ij} \delta_{ij},
\]

where \( N_{ij} \) is a number. Note well that, because of \( \delta_{ij} \), the above integral is zero unless \( i = j \). For \( i = j \), we get

\[
\int_{a}^{b} f_i^*(x)f_i(x)w(x)dx = N_{ii}.
\]
In general, the $N_{ii}$ are different for different values of $i$: for instance $N_{11}, N_{22}, N_{33}, \ldots$, may be different. It is critical to underscore the fact that one should not speak of orthogonal functions without specifying (1) the interval $[a, b]$ and (2) the weighting or density function. Please also note that $f_i^*(x)$ is the complex conjugate of $f_i(x)$.

**Warning:** * For real functions, $f_i^*(x) = f_i(x)$. This does not mean that one can write the definition without the complex conjugation of one of the functions.

* Some textbooks give definitions of orthogonal functions as $\int_a^b f_i(x)f_j(x)dx = \delta_{ij}$ without bothering to underscore that this definition is correct only if (1) all $f_i(x)$ are real functions and (2) $w(x) = 1$, and (3) $N_{ii} = 1$ for any value of $i$ from 1 to infinity.

**III. Examples of Orthogonal Functions**

3.1 $\sin(nx), n = 1 - \infty$, $\cos(n\theta), n = 0 - \infty$

In the applications of functions with a complex variable, we used the Moivre-Euler relation as follows:

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta, \quad i = \sqrt{-1}$$

From these relations we obtained

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

We then utilized the above expressions to evaluate integrals of the type

$$\int_0^{2\pi} \cos(nx)\sin(mx)dx = 0.$$
\[ \int_0^{2\pi} \cos(nx) \cos(mx) \, dx = \pi \delta_{nm} \text{ if both } n \text{ and } m \text{ are not zero.} \]

\[ = 2\pi \text{ if both } n = 0 \text{ and } m = 0 \]

\[ \int_0^{2\pi} \sin(nx) \sin(mx) \, dx = \pi \delta_{nm} \text{ if both } n \text{ and } m \text{ are non-zero. The integral is zero if } n = 0 \text{ or } m = 0. \]

A review of the definition of orthogonal functions, with \([a, b] = [0, 2\pi] \), \(w(x) = 1\) leads to the following results:

a) Functions \(\sin(nx), n \geq 1\), are orthogonal over the interval \([0, 2\pi]\) or any interval \([a, a + 2\pi]\) and with a weight function \(w(x) = 1\).

Note: In the above integrals, \(\sin(x) = \sin^*(x)\), and \(\cos(nx) = \cos^*(nx)\), given that the functions are real.

b) Functions \(\cos(nx), n = 0 - \infty\), are orthogonal over the interval \([0, 2\pi]\) or any interval \([a, a + 2\pi]\) and with a weighting function \(w(x) = 1\).

c) Functions \(\sin(nx), n = 1 - \infty\), and function \(\cos(nx), n = 0 - \infty\), are orthogonal over \([0, 2\pi]\) with \(w(x) = 1\)

Indeed, \(\int_0^{2\pi} \sin(nx) \cos(mx) \, dx = 0\)

\[ \int_0^{2\pi} \sin(nx) \sin(mx) \, dx = \pi \delta_{nm} \text{ [it is 0 if } n \text{ equals 0 or } m \text{ equals 0]} \]

\[ \int_0^{2\pi} \cos(nx) \cos(mx) \, dx = \pi \delta_{nm} \text{ if both } n \text{ and } m \text{ are not zero} \]

\[ = 2\pi \text{ if both } n = 0 \text{ and } m = 0 \]

3.2 \(e^{inx}, n = 0 - \infty\), are orthogonal functions over the interval \([0, 2\pi]\) or any interval \([a, a + 2\pi]\) and with a density function \(w(x) = 1\).

Caution \(e^{inx}\) is a complex function. Hence, the integral is \(\int_0^{2\pi} (e^{inx})' e^{inx} \, dx = 2\pi \delta_{nm}\)
or $\int_0^{2\pi} e^{-inx} e^{inx} \, dx = 2\pi\delta_{nm}$.

### 3.3 Orthogonal Polynomials

<table>
<thead>
<tr>
<th>Name of Polynomials</th>
<th>Weighting Function</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Legendre Polynomials</td>
<td>$w(x) = 1$</td>
<td>$[-1, +1]$</td>
</tr>
<tr>
<td>Laguerre Polynomials</td>
<td>$w(x) = e^{-x}$</td>
<td>$[0, \infty]$</td>
</tr>
<tr>
<td>Associated Laguerre</td>
<td>$w(x) = x^k e^{-x}$</td>
<td>$[0, \infty]$</td>
</tr>
<tr>
<td>Hermite Polynomials</td>
<td>$w(x) = e^{-x^2}$</td>
<td>$[-\infty, +\infty]$</td>
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</tbody>
</table>

### 3.4 Other Orthogonal Functions

There are many orthogonal functions that are not polynomials. Examples include $\sin(nx),\cos(nx)$, and the Bessel functions. The orthogonal polynomials in 3.3 and the Bessel functions are studied in Phys 411.

### IV. Normalization: Orthonormal Functions

In general, orthogonality leads to

$$\int_a^b f_i^*(x) f_j(x) w(x) \, dx = N_{ij} \delta_{ij}$$

So, for $i = j$, we get

$$\int_a^b f_n^*(x) f_n(x) w(x) \, dx = N_{nn} \quad (\delta_{nn} = 1)$$

If we define new functions $\varphi(x)$ such that

$$\varphi_n(x) = \frac{1}{\sqrt{N_{nn}}} f_n(x)$$

then

$$\int_a^b \varphi_n^*(x) \varphi_m(x) w(x) \, dx = \delta_{nm}.$$

Indeed, for $n \neq m$, the integral is zero. For $n = m$, it is
\[
\int_a^b \phi_n^*(x) \phi_m(x) w(x) \, dx = \frac{1}{N_{nm}} \int_a^b \phi_n^*(x) f_n(x) w(x) \, dx = \frac{N_{nm}}{N_{mn}} = 1
\]

When we normalize the orthogonal functions \( f_n(x) \), we get the orthonormal functions \( \phi_n(x) \). For orthonormal functions, the \( N_{nm} = 1 \). \( \frac{1}{\sqrt{\pi}} \sin(nx) \) and \( \frac{1}{2\pi} \) are orthonormal sets.

In the following section, we discuss some applications of orthogonal functions.

V. Applications of Orthogonal Functions

5.1 Evaluation of Integrals

The orthogonality of functions is utilized extensively in physics and related fields in the evaluation of integrals. Of course, in order to apply the orthogonality of some functions, one must know it correctly (i.e., interval, weight function, \( N_{nm} \)).

\[
\int_{-\pi}^{\pi} \cos(x) \, dx = \int_{-\pi}^{\pi} \cos(0x) \cdot \cos(1x) \, dx = \pi \delta_{01} = 0 ; \quad \int_0^{2\pi} \sin(nx) \cos(mx) \, dx = 0, \forall_n, \forall_m ; \quad \text{and} \quad \int_{-\pi}^{\pi} \cos^2(7x) \, dx = \pi, \int_{0}^{2\pi} \sin^2(8x) \, dx = \pi .
\]

The above simple examples illustrate the utter usefulness of orthogonality. This usefulness is much greater when working with orthogonal polynomials or functions in solving complicated or important problems.

5.2 Expansion of Functions in Terms of Orthogonal Functions

The power series expansion of a function is nothing else but the expansion of that function in terms of \( 1, x, x^2, x^3, x^4, \ldots x^n \). [These are not orthogonal functions]. The point is to recall that a function has to have derivatives of all orders to be expanded in terms of a \( 1, x, x^2, \ldots x^n \).
Similarly, to be expanded in terms of a given set of orthogonal or orthonormal functions, a function $\psi(x)$ or $G(x)$ has to meet certain criteria that depend on the orthogonal set.

Provided these conditions (criteria) are met, then $\psi(x)$ can be written (i.e., expanded) as follows:

$$\psi(x) = a_1 f_1(x) + a_2 f_2(x) + ... + a_n f_n(x) + ... = \sum_{n=1}^{\infty} a_n f_n(x)$$

A question that follows is: How do we find the $a_n$? The answer follows.

$$\psi(x) = \sum_{n=1}^{\infty} a_n f_n(x)$$

“Multiply” both sides by $\int_a^b f_i^*(x)w(x)dx$ and integrate.

$$\int_a^b f_i^*(x)\psi(x)w(x)dx = \sum_{n=1}^{\infty} a_n \int_a^b f_i^*(x)f_n(x)w(x)dx$$

We utilized the commutativity of the multiplication of functions and of the multiplication of a function by a constant (i.e., $a_n$). Using the very definition of orthogonality, we get

$$\int_a^b f_i^*(x)\psi(x)w(x)dx = \sum_{n=1}^{\infty} a_n N_{ln} \delta_{ln} = a_i N_{ll}$$

All the terms in $\sum_{n=1}^{\infty} a_n N_{ln} \delta_{ln}$ are zero, except the one for which $n \equiv l$. For $n = l$, $a_n \equiv a_l$. Using

$$\int_a^b f_i^*(x)\psi(x)w(x)dx = a_i N_{ll},$$

we get

$$a_i = \frac{1}{N_{ll}} \int_a^b f_i^*(x)\psi(x)w(x)dx$$

(To be derived and not memorized!)

From the preceding general case, we can derive the expansion coefficients (i.e., the $a_l$ or $a_n$), once we know, (a) that $\psi(x)$ meets the conditions for an expansion in terms of the orthogonal functions $f_n(x)$ and (b) the specifics of the orthogonality of the functions $f_n(x)$ [i.e., interval, $w(x)$, and the $N_{ll}$]. The next lecture on Fourier series provides a concrete example.